0908 Lecture 02

Tuesday, September 06, 2016 1:20 AM

Outline:

1) Basic facts about algebraic groups
2) Basic facts about group actions on quasi-projective schemes

① Basic facts about linear algebraic groups

k a field

G is a smooth affine group scheme$/k$

$$\left.\begin{array}{l} G \times G \xrightarrow{\mu} G \quad (\circlearrowleft i) \\ e : pt \longrightarrow G \\ \text{assoc, inverse, ident} \end{array}\right] \longleftrightarrow \left[\begin{array}{l} \text{commutative Hopf algebra } k[G] \\ k[G] \xrightarrow{\Delta} k[G] \otimes k[G] \\ k[G] \xrightarrow{\varepsilon} k \quad \circlearrowleft a \end{array}\right.$$

Remark: also characterized via functor of points $\rightsquigarrow$ functor
Rings $\longrightarrow$ groups

Rem: we also discuss group schemes sometimes

Eg $GL_n = \mathrm{Spec}\left(k[x_{ij}]\left[\frac{1}{\det}\right]\right)$

Eg $\mathbb{G}_m = \mathrm{Spec}\left(k[t^{\pm}]\right)$, $\Delta : k[t^{\pm}] \longrightarrow k[t_1^{\pm}] \otimes k[t_2^{\pm}]$
$t \longmapsto t_1 \otimes t_2$

split torus $(\mathbb{G}_m)^n$

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E.g. G is a torus if $G_{\bar{k}}$ is a torus

Deligne torus: Weil restriction of $(\mathbb{G}_m)_{\mathbb{C}}$ along $\mathrm{Spec}(\mathbb{C}) \xrightarrow{\pi} \mathrm{Spec}(\mathbb{R})$

$\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, $\mathbb{S}_{\mathbb{C}} = \mathbb{G}_m \times \mathbb{G}_m$

Weil restriction compatible w/ base change $\Big\}$ $\Rightarrow$ $\mathbb{S} = \mathrm{Spec}\left(\mathbb{C}[z^{\pm}, \bar{z}^{\pm}]^{\sigma}\right)$

takes disjoint unions to products

Generalization – A geometrically reduced finite commutative k-algebra, can Weil restrict $(\mathbb{G}_m)_A$ along $\mathrm{Spec}(A) \to \mathrm{Spec}\, k$, result is torus of rank = $\dim(A)$

$\hookrightarrow$ Note A action gives embedding $A^\times \hookrightarrow GL(A)$, max'l torus

Representations: Correct notion is comodule over $k[G]$:

$G \times V \to V \rightsquigarrow \mathrm{Sym}(V^*) \longrightarrow \mathrm{Sym}(V^*) \otimes k[G]$

$\rho : V^* \xrightarrow[k\text{-linear}]{} V^* \otimes k[G]$ $\rightsquigarrow$ (must lie in V^* because geometric points of must induce linear maps)

comodule axioms: 1) $V^* \longrightarrow V^* \otimes k[G] \xrightarrow{\epsilon} V^*$ is id.

2)
$$\begin{tikzcd} V^* \arrow[r, "\rho"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\circlearrowleft"] & V^* \otimes k[G] \arrow[d, "\rho \otimes 1"] \\ V^* \otimes k[G] \arrow[r] & V^* \otimes k[G] \otimes k[G] \end{tikzcd}$$

$$V^* \otimes k[G] \xrightarrow[1\otimes\Delta]{} V^* \otimes k[G] \otimes k[G]$$

<u>Ex:</u> category of $\mathbb{G}_m$-modules is equivalent to graded vector spaces over any field, of any dimension (for a split torus M-graded)

<u>Ex:</u> Representations of Deligne-torus are real Hodge structures

<u>Prop:</u> every representation is a union of finite sub rep.s

<u>Pf:</u> write $\rho(v) = \sum v_i \otimes e_i$ basis for $k[G]$

define $\Delta(e_i) = \sum r_{ijk}\, e_j \otimes e_k$, $r_{ijk} \in k$

$$\overset{\text{associativity}}{\Longrightarrow} \sum_{ijk} r_{ijk} (v_i \otimes e_j \otimes e_k) = \sum_k \rho(v_k) \otimes e_k$$

$$\Longrightarrow \rho(v_k) = \sum_{ij} r_{ijk} (v_i \otimes e_j) \Longrightarrow \{v_i\} \text{ span a submodule containing } v$$

$\blacksquare$

<u>Thm: Main structural properties</u>

a) $G \hookrightarrow GL_n$ for some k (Find a finite subrep of regular rep. containing set of gen's of $k[G]$)

b) Jordan decomposition for any $g \in G(\bar{k})$, $g = g_{ss} \cdot g_u$

semistable unipotent and they commute

$g \in G(k)$, $g = g_{ss} \cdot g_u$

semistable, unipotent, and they commute

Functorial

defined in some (and all) linear embeddings

c) $\exists$ maximal connected solvable $B \hookrightarrow G \Leftrightarrow G/B$ is projective

(in fact every geom. point lies in one) over $\bar{k}$, unique up to conjugacy (Remark on parabolics)

d) $\exists$ maximal torus $T \hookrightarrow G$, $T_{\bar{k}} \hookrightarrow G_{\bar{k}}$ is maximal as well, and unique up to conj.

e) $\exists!$ maximal unipotent conn. <u>normal</u> subgp $R_u(G) \hookrightarrow G \twoheadrightarrow H$

"unip. radical"

↳ <u>Defn:</u> G reductive if $R_u(G) = 1$

f) Reductive $\Leftrightarrow$ linear reductive (meaning the category of $k[G]$ comodules has no higher ext's) <u>in char 0</u>

isogenous to torus $\times$ semisimple

In char $p>0$: Nagata, conn component $G_0 \cong (\mathbb{G}_m)^n$ and $|G/G_0|$ prime to p

Finite dimensional lin. reps in char 0 are classified by their characters $\rightsquigarrow$ irreps by highest weight.

prime to p

Finite dimensional lin. reps in char 0 are classified by their characters $\rightsquigarrow$ ~~kk~~ irreps by highest weight.

Remark: If G is split-reductive any parabolic $P \subset G$ is <u>standard</u> meaning there is some 1PS $\lambda : \mathbb{G}_m \hookrightarrow G$ and a linear embedding such that

$$P = G \cap \{\text{block upper triangular matrices}\}$$

w.r.t. corresponding filtration

Rem: over $\bar{k}$, G is rational as a variety

Ref's: Borel, linear alg. groups
Conrad, beginning of reductive gp. schemes
Milne, algebraic groups

Group actions on schemes:

ex1 $X \to G\times X \longrightarrow X$

Def: $G\times X \xrightarrow{\sigma} X$ satisfying axioms $\begin{cases} 1) \text{ identity} \\ 2) \text{ associativity} \end{cases}$

X a scheme

Important map $G\times X \longrightarrow X\times X$
$(g,x) \longmapsto (gx, x)$

(or more generally $\psi_f : G\times T \to X\times T$ for any T-point $f: T \to X$)

e.g. for a k-point, get $\psi_x : G \to X$ ⇝ its image is the orbit, fiber over x is $\mathrm{Stab}_x \subset G$, a sub gp

↳ more generally

$$\begin{tikzcd} \mathrm{Stab}_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & G\times T \arrow[d, "\psi_f"] \\ T \arrow[r, "{(f,1)}", "\text{diag}"'] & X\times T \end{tikzcd}$$

get inertia group, I_X ⇝ a group scheme over X giving stabilizer of each point

Rem: Fiber dimension of $I_X \longrightarrow X$ is upper semi-continuous because

Rem: fiber dimension of $I_X \longrightarrow X$ is upper semi-continuous because it is a locally finite type group scheme

Most concrete: Actions on affine scheme $X = \operatorname{Spec} A$

Equivalent to giving $k[G]$-comodule structure on A such that $\rho: A \to A \otimes k[G]$ is a map of algebras

Lemma: If A is finite type$/k$, there is a linear rep. V and an embedding $\operatorname{Spec} A \hookrightarrow V$ which is G-equivariant

Pf (idea): V is the dual of a finite dim'l invariant sub-rep of A which generates A as an algebra. $\square$

Eg: an affine scheme w/ T action, where T is split torus, is equiv. to an M-graded algebra, where M is char lattice

↳ $\boxed{\text{useful fact:}}$ every invariant ideal is gen. by homogeneous elements.

Remark: Matsushima's theorem: If G is a reductive algebraic group and $H \subset G$ an algebraic subgroup, then

H reductive $\iff$ G/H is affine

H reductive $\Leftrightarrow$ G/H is affine
always q-proj.

(Alper has proves the same result for lin. reductive fppf group schemes over arbitrary base)

Then choose $G \hookrightarrow GL_n$, then GL_n acts transitively on $X = GL_n/G = \operatorname{Spec} A$
by above, have closed imm. $\operatorname{Spec}(A) \hookrightarrow V$ for some GL_n-rep V
$\Longrightarrow$ $\forall$ reductive G, $\exists$ a rep of GL_n and a $v \in V$ w/ closed orbit such that $G = \operatorname{stab}(v)$
This statement (except for v having closed orbit) is true for any G.

<u>Next level of complexity</u>: V linear rep of G $\rightsquigarrow$ action on $\mathbb{P}(V)$

Consider $X \hookrightarrow \mathbb{P}(V)$ which is equivariant ($G \times X \longrightarrow \mathbb{P}(V)$ factors through X, in which case it does so uniquely and induces an action on X)
(Rem: we will see later that this is true for any normal proj. variety)

<u>Eg</u>: Homogeneous spaces G/H are always G-quasi-projective

<u>Existence of quotients</u>: We don't quite have the technology yet, but there is a very general

<u>Thm</u>: If $G \times X \hookrightarrow X \times X$ is a closed immersion, then the sheaf X/G is an algebraic space

<u>Thm</u>: every algebraic space has an affine open subspace

$\Longrightarrow$ for free actions, you can at least form a q-proj. quotient for some open subscheme

Special results for T-varieties:

<u>Thm (Sumihiro)</u> Let $X \hookrightarrow \mathbb{P}(V)$ be a T-equivariant q-proj. subvariety, then for any point $x \in X$, $\exists$ a T-equivariant affine open containing x.

<u>Proof</u>: • case $\overset{k=\bar{k}}{X} \hookrightarrow \mathbb{P}(V)$ closed: use standard open $\mathbb{P}(V)_f \cap X$ where $f \in V^*$ is eigenvect.

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- case $k = \bar{k}$, $X \hookrightarrow \mathbb{P}(V)$ loc. closed: consider closure, then have

  T-inv. affine variety $Y \supset X \cap Y \ni x$

  Show that you can find an eigenvector $f \in \mathcal{O}(Y)$ which vanishes on $Y \setminus X \cap Y$ and does not vanish at x

- Both steps use standard opens for some functions f. Such a function is defined over some finite Galois extension of k

  $\Longrightarrow$ taking the intersection of the Galois orbits of U gives an aff open subset defined over k

$\blacksquare$

Fixed loci and Białynicki-Birula theorem

Thm: Let T act on $X \hookrightarrow \mathbb{P}(V)$, then the functor $R \longmapsto \mathrm{Map}_T(\mathrm{Spec}\, R, X)$ is representable by a closed subscheme of X

trivial T-action

It is smooth if X is smooth.

Pf: can reduce to affine case using Sumihiro, and suffices to assume $k = \bar{k}$, so T is lin. reductive

A is M-graded algebra, let $I \subset A$ be the ideal gen. by $\bigoplus_{\chi \in M \setminus \{0\}} A_\chi$

Then $Z = \operatorname{Spec}(A/I)$ represents the functor. It is smooth because can show that $T_x Z = (T_x X)^T$; and can lift non-invariant elements of m_x/m_x^2 to elements of m_x which define Z as a transverse intersection at x.

$\blacksquare$

<u>Thm (Białynicki-Birula)</u>: $X \hookrightarrow \mathbb{P}(V)$ is $\mathbb{G}_m$-quproj. then the functor

1) $R \longmapsto \operatorname{Map}_{\mathbb{G}_m}(\operatorname{Spec}(R) \times \mathbb{A}^1, X)$ is representable by a scheme

2) the restriction $\operatorname{Map}_{\mathbb{G}_m}(\operatorname{Spec}(R) \times \mathbb{A}^1, X) \xrightarrow{i} \operatorname{Map}(\operatorname{Spec}(R) \times \{1\}, X)$ is an embedding.

3) if X is smooth, then $\operatorname{Map}_{\mathbb{G}_m}(\operatorname{Spec}(R) \times \mathbb{A}^1, X) \xrightarrow{\pi} \operatorname{Map}_{\mathbb{G}_m}(\operatorname{Spec}(R) \times \{0\}, X)$ is a locally trivial bundle of affine spaces

is a locally trivial bundle of affine spaces

<u>Pf:</u> (Idea) Similar to last, can reduce to the affine case. In that case $\mathrm{Spec}(A/I^+)$ represents the functor, where I^+ is the ideal gen. by elements of positive weight. $\boxtimes$

<u>Ex:</u> $\mathbb{P}(V)$ with a linear action of $\mathbb{G}_m$

$\rightsquigarrow$ choose eigenbasis, i.e. coordinates s.t. $\mathbb{G}_m$-action is

$[t^{a_0}z_0 : \cdots : t^{a_n}z_n]$ with $a_0 \leq \cdots \leq a_n$

$\rightarrow$ then fixed loci are eigenspaces for each a

$\rightarrow$ BB strata look like $[0:\cdots 0:1:*:*:\cdots]$